Dynamical Calculations and Regge Trajectories in the Presence of Spin*

ERNEST ABERS AND FREDRIK ZACHARIASEN
California Institute of Technology, Pasadena, California
(Received 28 May 1964)

Some new features of the Regge trajectory structure of scattering amplitudes for spinning particles, associated with the existence of a nonsense channel, are described. The restrictions imposed by smoothness in angular momentum on the input forces to dynamical calculations are then discussed. Some popular versions of the N/D method are not, in general, consistent with these restrictions, but can be made so by special choice of subtractions. We illustrate these ideas in detail for the scalar-vector scattering amplitude. The characteristic new feature is a trajectory α_0 which goes to zero in the limit of small coupling constant; it emerges naturally from the N/D method with our special subtractions. The dynamics of particles lying on this trajectory are curious, and may reflect a weakness of the N/D approximation rather than physics, since strong forces (large coupling constants) are not needed to bring it up to a physical value of the angular momentum. As an example, we perform a simple bootstrap calculation for pseudoscalar mesons, assuming that they lie on such a trajectory.

I. INTRODUCTORY REMARKS

THE advent of the bootstrap hypothesis[1] has stimulated a great deal of interest in dynamical calculations with strongly interacting particles during the past few years. Most of these calculations have been carried out by means of some version of the "N-over-D" method,[2] which in practice amounts to "souping up" perturbation theory to a point where it is capable of giving bound states or resonances. Our purpose here is to discuss the use of these methods in problems in which the external particles have spin.

The generalization of the N/D technique and its relatives to the scattering of spinning particles is neither entirely obvious nor free from ambiguities. There are some properties which are directly associated with the existence of nonsense channels and which do not have spinless analogs; it is primarily these which we intend to explore here.

We wish to begin by digressing a bit from our principal topic, to make it clear that although we shall discuss properties of N/D type methods, we do not mean to imply that we feel these methods to be particularly reliable quantitatively. Nevertheless, they have some interesting new features when the external particles have spin; we shall discover some restrictions on the use of these methods which are not generally known.

N/D type methods are always dependent on assuming input forces, which in practice can, at present, hardly be much more than those provided by single particle or resonance exchange. The details of shorter range forces, to which all too many calculations are sensitive, must be lumped into phenomenological parameters such as cutoffs; and while it is true that cutoffs can be related to properties of Regge trajectories in the crossed channels, this fact is really only of psychological value at present, since little is known about those trajectories. A way of minimizing these uncertainties may be the bootstrapping of entire Regge trajectories; this procedure in some sense allows the self-consistent determination of short- as well as long-range forces. However, explicit methods for doing this are complicated and, as far as we know, have not yet been exploited very much, although several are being investigated.[3] If these methods will be used for problems with spin, considerations analogous to the ones we shall discuss will probably have to be taken into account there also. We shall not pursue them further.

In addition to these uncertainties about the choice of input forces in N/D methods, practical limitations confine calculations to only a very few channels at a time. It was originally hoped that only the channels with nearby thresholds are important; but the apparent existence of a host of new particles spread all over the mass spectrum, which was not known a few years ago, makes it likely that in most cases there are too many nearby thresholds, unless one uses a rather implausible definition of "nearby." Furthermore, the fact that the present state of the art does not *ever* allow the inclusion of channels with more than two particles, no matter how "nearby," is additional cause for apprehension.

Finally, a truly dynamical theory requires one to treat the external particles in a scattering process as composite, which in practice means one should include certain higher order processes contributing to the structure of the external particles. The N/D method fails to do this, and this failure, perhaps especially in the case of spin, as we shall see later, may lead to consequences of dubious validity.

For all these reasons we feel it is highly optimistic to expect earth-shaking results from practical calculations with any of the standard methods. It is therefore difficult to get excited about the relative merits of this or that version of the N/D method, or this or that simplifying assumption made within its framework. Consequently, we shall remark on the complications produced by spin

* Work supported in part by the U. S. Atomic Energy Commission and the Alfred P. Sloan Foundation.

[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).

[2] G. F. Chew and S. C. Mandelstam, Phys. Rev. **19**, 467 (1960).

[3] G. F. Chew, Phys. Rev. **129**, 2363 (1963). S. C. Frautschi, P. E. Kaus, and F. Zachariasen, *ibid.* **133**, B1607 (1964).

in some of the standard techniques without arguing in favor of one or the other. Specifically, we shall discuss the full N/D integral equations, and various symmetrized versions of the determinantal method (at least the customary first-order approximation to it[4]). In practice, these methods have all produced similar results when applied to the same dynamical problem.

The bootstrap hypothesis is essentially the hypothesis that there are no elementary particles, or perhaps equivalently, that there are no uncalculable parameters. This assumption must be translated into mathematical terms so that it may be used as a boundary condition, so to speak, on dynamical calculations. Unfortunately, we do not know with certainty exactly what these mathematical terms should be. In some theories like potential theory, the number of elementary particles in a given channel is related by Levinson's theorem[5] to the change in phase shift between threshold and infinity for that channel, and to the number of dynamical bound states. The same relation, taken over into relativistic particle physics, suggests that the boundary condition to be imposed here is the absence of CDD poles in the D function.[6] It has also been suggested that the restriction that all particles lie on Regge trajectories is equivalent to the statement that they are not elementary.[1] In ordinary potential theory, these two statements seem to be the same. In particle physics the connection between all these statements is not clear; any mathematical property which guarantees the calculability, within our present theoretical understanding, of the properties of a particle may be used as a definition of its dynamical nature. It is not even clear that Reggeism is such a property, since there is evidence that in certain theories a particle with apparently undetermined mass and coupling constant lies on a Regge trajectory.[7] Nevertheless, we shall accept it by default as our criterion.

It is, of course, open to argument whether or not one should try to force on a method which is so approximate a general feature like smoothness in the angular momentum. The alternatives to doing this, however, seem in practice to be limited to the use of particular field theory prescriptions, which dictate specified Kronecker deltas in the angular momentum. Insofar as one would like to pretend to be calculating within the framework of a purely dynamical theory, in contrast to a field theory, these Kronecker deltas are embarrassing, since it is basically only in them that the dynamical theory and the field theory differ.

Our purpose, then, is to discuss dynamical calculations of the N/D type when the external particles have spin, with the limitation that the partial-wave amplitudes be purely Regge-like; by which we mean that they shall be analytic functions of the angular momentum with only moving (Regge) poles.

We shall generally limit ourselves to talking about the elastic scattering of a vector particle by a scalar or pseudoscalar particle, but it should be evident that our remarks apply to any problem having a nonsense channel, and we shall in fact occasionally refer to vector-spinor scattering as an additional example. We shall, in the vector-scalar case, be primarily interested in the behavior of the partial-wave amplitudes near $j=0$ (j is the total angular momentum), since it is primarily there that problems specifically connected with the existence of spin occur. In order to study this, we shall set up the problem at small but nonzero j, and then continue to $j=0$, assuming, because of the required Regge-like behavior, that the amplitude at $j=0$ can be obtained by such continuation, and that it contains no terms like δ_{j0}, for example.

We shall begin with a brief discussion of what general form we expect the amplitudes to have and what properties should hold as a result of the assumption of pure Reggeism. Next we turn to a discussion of input forces, and show how to choose them in a way consistent with the required Regge nature of the partial-wave amplitudes. In Sec. IV we shall set up the N/D equations with this input, and carry out the continuation to $j=0$. At this value of j (in problems where the physical angular momentum is integral), the nonsense channels cease to be coupled to the sense channels, so that in the vector-scalar case, for example, a two-channel problem is suddenly reduced to a single-channel one, for the ${}^3P_0$ amplitude alone. We then discuss the equivalent one-channel problem at $j=0$, consistent with smooth continuation from finite j. We shall analyze the coupled integral equation N/D method in detail, but also remark on the analogous situation in the determinantal and pole approximations.

A basic difference between the problems with and without spin is the existence, in the case of spin, of a Regge trajectory which, in the limit of zero coupling, approaches zero, a physical value of j. Dynamical particles may lie on this trajectory as well as on the others, which are analogous to those which in the spinless case start (in the zero-coupling limit) from $j=-1, -2, \cdots$, etc. A simple approximation is suggested in Sec. V for finding the $j=0$ states lying on this trajectory, which is the analog of the first-order determinantal approximation for states lying on the familiar trajectories. This approximation is, of course, not epected to be particularly believable. Indeed, the states obtained in this way may very well have nothing to do with physics, for they have their apparent origin in the properties of the nonsense channel, which, as we shall see, are relevant only because of singularities near $j=0$ in the input force. These singularities, in turn, are directly related to the fact that even though one purports to be studying a dynamical theory, the usual N/D approximation considers the external particles to be elementary.

[4] M. Baker, Ann. Phys. (N. Y.) 4, 271 (1958).
[5] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. 25, No. 9 (1949).
[6] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin and Company, Inc., New York, 1961), Chap. 11.
[7] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964).

Nevertheless the approximation has many amusing new features, and we do not know how they are reflected in a "more exact" calculation. It is illustrated by the bootstrapping of a pseudoscalar multiplet by pseudoscalar-vector scattering, in both SU(2) and SU(3) models.

II. GENERAL FORM OF A REGGE-LIKE AMPLITUDE

Let us review briefly the anticipated behavior of Regge-like partial-wave amplitudes[8] $t^j(s)$ in the angular momentum plane. Here s is center-of-mass frame energy squared. The $t^j(s)$ are matrix analytic functions of j except for energy-dependent poles at $j=\alpha(s)$,[9] so they should not contain any terms like, for example, δ_{j0} or δ_{j1}.

If $t^j(s)$ is the amplitude for scattering of two spinless particles into two spinless particles, it will be a single scalar function; for a general problem, $t^j(s)$ is a matrix whose dimension N is the number of coupled channels. Thus for vector-scalar scattering, $t^j(s)$ is a 2×2 matrix for parity $(-1)^j$, while for parity $-(-1)^j$ it is a single scalar function. For vector-spinor scattering, $t^j(s)$ is a 3×3 matrix for each parity, and so forth.

The assumption that $t^j(s)$ is purely Regge-like allows us to write it as a sum of terms like

$$[1/(j-\alpha(s))]\beta(s)\,, \tag{2.1}$$

plus the cut terms which we are ignoring, plus the background term to fix up the asymptotic behavior for large j. The residues $\beta(s)$ are matrices, which are known to factor[10]; i.e.,

$$\beta_{\mu\nu}(s)=\eta_\mu(s)\eta_\nu(s)\,, \tag{2.2}$$

for some N functions η_μ. In the spinless problem, $N=1$, and factoring is no restriction, but it is one of the principal sources of additional structure in the spinning case. Furthermore, if there is a nonsense channel present at $j=n$,[11] where n is a physical value of the angular momentum, the matrix elements $\beta_{\mu\nu}$ for which μ is a sense index and ν is a nonsense index must vanish at $j=n$ like $(j-n)^{1/2}$, so that the sensible channels are completely decoupled from the nonsense ones. It will be convenient to write these square-root factors explicitly, to avoid introducing kinematical branch points in j into the $\beta_{\mu\nu}$.

We shall discuss only theories in which the scattering is characterized by some collection of coupling strengths, which for simplicity we consider to be one number g^2;[12] and in which the scattering amplitude vanishes as $g^2\to0$. The trajectory and residue functions may then be identified by comparing their $g^2\to0$ limits with perturbation theory. We expect, for small g^2,

$$\beta\to B+O(g^4)\,, \tag{2.3}$$

where B is of order g^2. In the same limit, each trajectory α will approach some constant which is independent of g^2.

For example, in the spinless case we expect from perturbation theory that $t^j(s)$ contains a term of order g^2 of the form

$$g^2f(s)Q_j(x(s))\,, \tag{2.4}$$

where $f(s)$ and $x(s)$ are simple functions of s which can be computed from the mass and spin of the exchanged particle. For fixed x, $Q_j(x)$ has poles at each negative integer $-n$, of the form

$$P_{n-1}(x)/(j+n)\,. \tag{2.5}$$

We might guess, then, in analogy with potential theory, that relativistic spinless scattering has a Regge trajectory $\alpha_n(s)$ for each positive integer n, with a residue function $\beta_n(s)$, such that

$$\begin{aligned}&\alpha_n(s)\to-n\,,\\ &\beta_n(s)\to g^2f(s)P_{n-1}(x(s))=B_n(s)\,,\end{aligned} \tag{2.6}$$

as $g^2\to0$. It could also be the case, of course, that there is more than one trajectory associated with each integer.

For the time being, let us normalize our amplitudes so that the eigenvalues of $t^j(s)$ are of the form $e^{i\delta}\sin\delta$. Then, in the spinless problem, from the unitarity condition (for real j)

$$\mathrm{Im}t^j(s)=t^j(s)t^j(s)^*\,, \tag{2.7}$$

we can conclude that to order g^2,

$$\mathrm{Im}\alpha_n(s)=B_n(s)\,. \tag{2.8}$$

When the external particles have spin, several new features appear. Let us look at the example of vector-scalar scattering. For the states with parity $(-1)^j$, $t^j(s)$ is a two-dimensional matrix, since the orbital angular momentum l may be equal to $j\pm1$. At $j=0$, the sense channel becomes pure ${}^3P_0$, while the nonsense channel becomes pure ${}^3A_0$. (A means $l=-1$, and stands for "absent.") The Q functions occurring in the g^2 terms arise from spatial angular integrations, and therefore are indexed by l rather than j; so the nonsense channel partial-wave amplitudes to order g^2 will have a term proportional to $Q_{j-1}(x)$, and therefore a pole at $j=0$. (The sense channel amplitudes will be finite there.) We thus anticipate the presence of at least one new trajectory

[8] The principal results in this and the succeeding section are mostly already known to some people; they are contained, at least implicitly, in Ref. 7, and in S. Mandelstam, Nuovo Cimento **30**, 1113 (1963).

[9] There are also generally cuts in the j plane. S. Mandelstam, Nuovo Cimento **30**, 1124 (1963). None of the approximate methods we discuss here are sufficiently detailed to produce these cuts, however, so we shall ignore them.

[10] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **8**, 343 (1962). M. Gell-Mann, *ibid.* **8**, 263 (1962).

[11] M. Gell-Mann, *Proceedings of the 1962 Annual International Conference on High Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 533.

[12] Such theories may not include a real bootstrap theory, which has no arbitrary parameters, and the limit $g^2\to0$ has no meaning. The methods we discuss, however, all assume to begin with that g^2 is an ordinary coupling parameter, and after calculating an amplitude which depends on g^2, determine it by a self-consistency requirement.

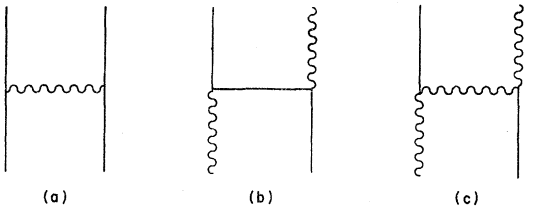

FIG. 1. Input force diagrams for (a) spin-0–spin-0 scattering, and (b) and (c) spin-0-spin-1 scattering. Wavy lines are vector mesons, solid lines are (pseudo) scalar mesons.

$\alpha_0(s)$, such that as $g^2 \to 0$,

$$\alpha_0(s) \to 0+O(g^2), \tag{2.9}$$

and the associated residue matrix looks like

$$\beta \to \begin{pmatrix} 0 & j^{1/2}B_x \\ j^{1/2}B_x & B_2 \end{pmatrix} + O(g^4). \tag{2.10}$$

Here, the 11 element of β is the sense channel amplitude, the 22 element the nonsense channel amplitude, and we have written the $j^{1/2}$ factor in the crossed terms explicitly. These statements are illustrated by the example quoted in the next section. It should also be noted that aside from kinematical factors, B_x and B_2 are always simply polynomials in s. β_x and β_2, of course, may have dynamical cuts.

As in the spinless case, there are also trajectories approaching each negative integer in the perturbation limit, but now, since the residues of the poles in the g^2 terms do not in general factor, there must be at least two trajectories going to each negative integer. In general, for an N-dimensional scattering matrix, we can anticipate at least N trajectories at each negative integer, while the number of trajectories at each physical j is at least equal to the number of nonsense channels there.

Let us look more closely at the new trajectory $\alpha_0(s)$. We may write its contribution to the entire amplitude as

$$\frac{1}{j-\alpha_0}\begin{pmatrix} \beta_1 & j^{1/2}\beta_x \\ j^{1/2}\beta_x & \beta_2 \end{pmatrix}. \tag{2.11}$$

Note that $\beta_1=\alpha\beta_x^2/\beta_2$ because of the factoring condition, so that β_1 is of order g^4, in agreement with (2.10). Therefore, the contribution of this trajectory to $t^j(s)$ is

$$\frac{1}{j}\begin{pmatrix} 0 & j^{1/2}B_x \\ j^{1/2}B_x & B_2 \end{pmatrix} + O(g^4), \tag{2.12}$$

if $j\neq 0$. If $j=0$, however, there is a contribution of order g^2 to the sense-sense amplitude. $t_{11}{}^0$ contains

$$\frac{\alpha\beta_x^2/\beta_2}{-\alpha} = -\frac{\beta_x^2}{\beta_2} = -\frac{B_x^2}{B_2} + O(g^4). \tag{2.13}$$

There is thus, to order g^2, a term $-(B_x^2/B_2)\delta_{j0}$ in the sense-sense amplitude, the existence of which will be necessary if the transition to $j=0$ is to be smooth in j. It is important to emphasize that the δ_{j0} occurs here only to order g^2; to higher orders, it looks like $-\alpha_0/(j-\alpha_0)$ and is in fact analytic in j.

We may again find $\mathrm{Im}\alpha_0(s)$ to second order in g^2 by unitarity:

$$\mathrm{Im}\alpha_0(s) = B_2(s) + O(g^4). \tag{2.14}$$

III. INPUT FORCES

Terms proportional to δ_{j0}, $\delta_{j1}, \cdots$, etc., in the partial-wave amplitude result from polynomials, coming from subtractions, in the momentum transfer in the full scattering amplitude. The rule for a Regge-like amplitude, "smoothness in j," requires us to discard all δ_{j0}, etc., terms; it therefore also tells us how to treat the polynomials in the momentum transfer.

In the calculational schemes we are discussing, one chooses an input force, or Born term, and then computes an "output" amplitude by N/D or some similar method. As an example, consider the scattering of two scalar particles of mass μ, where the force is the exchange of a vector particle of mass m [see Fig. 1(a)]. Let g be the vector-scalar-scalar coupling constant. The input scattering amplitude, which is the Born term for this process, is proportional to

$$B(s,t) = \frac{g^2}{8\pi^2}\left[\frac{2s+m^2-4\mu^2}{t-m^2} + \text{polynomial in } t\right]. \tag{3.1}$$

The input partial-wave amplitude, normalized as in (2.8), is

$$B^j(s) = -\frac{g^2}{4\pi}\frac{q}{W}\left[\frac{2s-4\mu^2+m^2}{4q^2}\right]Q_j(z) + \text{terms in } \delta_{l0}, \delta_{l1}, \cdots. \tag{3.2}$$

Here, q and W are the center-of-mass frame momentum of each particle and the total energy, respectively; $s=W^2$ and $z=1+m^2/(2q^2)$. Since there are only a finite number of Kronecker delta terms, the correct input for sufficiently high j is evidently just the term containing $Q_j(z)$. But $B^j(s)$ must produce a partial-wave amplitude $t^j(s)$ which is analytic in j and has the correct $j\to\infty$ limit to permit a Sommerfeld-Watson transformation. Knowledge of $t^j(s)$ for an infinite collection of sufficiently high j, plus this boundary condition at infinity, prescribes $t^j(s)$ unambiguously for all j. For our spinless example, the prescription is clear: Simply leave out all δ_{j0}, δ_{j1}, etc., terms from the input force $B^j(s)$.

It is important to remark that the Regge-like force for this spinless example is *not* the Born term obtained by computing the Feynman diagram for vector meson exchange with the usual conserved-current coupling. This amplitude contains, in addition to the Q_j term, a term $(g^2/8\pi)q/W\delta_{j0}$, in disagreement with our prescription. The input force for a Regge-like theory, as is a well-

known but frequently forgotten fact,[13] cannot in general be obtained by projecting a Feynman diagram at some fixed value of j.

To illustrate the situation in the presence of spin, let us refer again to our vector-scalar scattering example, with the same masses and couplings as above; the input force comes from scalar exchange [see Fig. 1(b)]. We avoid irrelevant complications by assuming both particles to be stable ($m<2\mu$). We use parity-conserving helicity amplitudes[14] and then write B^j as a 2×2 matrix for the parity $(-1)^j$ states, denoting by index 1 the longitudinal (helicity=0) states and by index 2 the transverse (helicity=±1) states. This agrees with our notation in Sec. II, since near $j=0$ index 1 is sense and index 2 is nonsense. Again normalizing as in (2.8), we can find the input for all j to order g^2 simply by projecting the Feynman diagram at high j:

$$\begin{aligned}
B_{11}{}^j(s)&=-\frac{g^2}{4\pi}\frac{q}{m^2W}(E_s+xE_v)^2Q_j(x)\,,\\
B_{12}{}^j(s)&=B_x{}^j(s)=B_{21}{}^j(s)=-\frac{g^2}{4\pi}\frac{q}{mW}\left(\frac{1}{j+1}\right)^{1/2}\\
&\qquad\times(E_s+xE_v)[Q_{j-1}(x)-xQ_j(x)]\,,\\
B_{22}{}^j(s)&=\frac{g^2}{2\pi}\frac{q}{W}\frac{1}{j+1}\\
&\qquad\times[(j+1-jx^2)Q_j(x)-xQ_{j-1}(x)]\,.
\end{aligned}\tag{3.3}$$

Here, E_s and E_v are the c.m. energies of the scalar and vector particles, respectively, and

$$x=1+(2m^2+\mu^2-s)/2q^2\,.\tag{3.4}$$

Near $j=0$, Eq. (3.3) becomes

$$\begin{aligned}
B_{11}{}^j(s)&\approx-\frac{g^2}{4\pi}\frac{q}{Wm^2}(E_s+xE_v)^2Q_0(x)\equiv\frac{1}{qW}b_1(s)\,,\\
B_x{}^j(s)&\approx-\frac{g^2}{4\pi}\frac{q}{mW}(E_s+xE_v)/j^{1/2}\equiv\frac{1}{qs}\left(\frac{b_x(s)}{j}\right),\\
B_{22}{}^j(s)&\approx-\frac{g^2}{4\pi}\frac{q}{W}\frac{x}{j}\equiv\frac{1}{qWs}\left(\frac{b_2(s)}{j}\right).
\end{aligned}\tag{3.5}$$

Notice that with the kinematical singularities factored out explicitly as indicated, b_x and b_2 are polynomials in s.

Because of the singularities at $j=0$, one cannot simply use this input there. Instead, it must be used to calculate the amplitudes at small but nonzero j, and the result continued to $j=0$. As was made evident in Sec. II, the resulting g^2 term in the 11 amplitude at $j=0$ will *not* be simply $(1/Wq)b_1$ as defined in Eq. (3.5) above, but rather $(b_1-b_x{}^2/b_2)/qW$; while for all other j, this quantity is just $(1/qW)b_1$. It is encouraging to note that this behavior is consistent with the required threshold behavior. For at $j\neq0$, all the components of $t^j(s)$ should behave like q^{2j-1} at threshold. Exactly at $j=0$, however, $t_{11}{}^0(s)$ should be like q^3, not q^{-1}, since it is a pure ${}^3P_0$ amplitude. Now b_1, b_x, and b_2, with our definitions, all behave like constants, but one can check that the combination $b_1-b_x{}^2/b_2$ is like q^4. We shall show in the next section that this is a general property.

It is interesting to notice that a given set of Feynman graphs will contribute to $t_{11}{}^0(s)$ certain explicit δ_{j0} terms, as well as the Q_j terms indicated in (3.3). These δ_{j0} terms are also, of course, arranged to fix up the threshold behavior in $b_{11}{}^0$, just as $[-(b_x{}^2/b_2)/qW]\delta_{j0}$ does. However, for the scalar-vector case, the Feynman graphs will not give precisely *this* δ_{j0} term. In contrast, the lowest order Feynman diagrams for spinor-vector scattering with spinor exchange, with conserved current coupling, produces exactly the analog of $-(1/qW)(b_x{}^2/b_2)\delta_{j0}$.[7]

Finally, we remark that if instead of choosing essentially the Born approximation as input, one tried to include some higher order terms in g^2 by using a Reggeized input, one would of course avoid the singularities at $j=0$ in the nonsense and mixed amplitudes. However, as we remarked in the introduction, to do this in a completely self-consistent manner would involve very complicated techniques which we do not want to discuss here.[3] Nevertheless, without attempting a really correct treatment of a Regge input, we may remark on a sort of schematic Reggeization of the input which has occasionally been used. This amounts simply to multiplying the contribution of each Feynman graph by a factor $s^{\alpha(t)}$, which is supposed to represent partially the effect of Reggeizing the input. Most simply, the trajectory $\alpha(t)$ is chosen as a straight line, say, $\alpha(t)=\alpha(0)+\alpha'(0)t$. The parameters are either guessed from some rough analysis of experiments or they may be determined self-consistently in the bootstrap sense.[15] The self-consistency is, of course, illusory, since the output will not be a straight line, but it still may be a reasonable approximation at low energies when the input trajectory is needed only over a small range of t. With this modification of the input, the polynomials in t appearing in a given lowest order Feynman diagram become smoothed out in j, instead of producing δ_{j0}, δ_{j1}, etc. For example, a term like

$$\int_{-1}^{1}dx\,P_j(x)f(s)=2f(s)\delta_{j0}\tag{3.6}$$

[13] Among the many calculations in which this error has been made are: F. Zachariasen and C. Zemach, Phys, Rev. **128**, 849 (1962); E. Abers and C. Zemach, *ibid.* **131**, 2305 (1963); E. Abers, Phys. Rev. Letters **12**, 55 (1964). In the last example, $\pi-\omega$ scattering with ρ-meson exchange, a Regge-like input is an attractive force in the 2^- channels, but a repulsive force in the 1^+, in contradiction to what is obtained there by projecting Feynman diagrams.

[14] See Ref. 7, Appendix A.

[15] D. Y. Wong, Phys. Rev. **126**, 1220 (1962). We are indebted to Dr. R. C. Arnold for calling our attention to this way of modifying the input.

becomes replaced by something like

$$\int_{-1}^{1} dx\, P_j(x) s^{\epsilon x} f(s) = 2f(s) i_j(\epsilon \ln s), \tag{3.7}$$

which is smooth in j. Here, i represents a modified spherical Bessel function of the first kind. In addition, the $1/j$ factors coming from Q_{j-1} in the partial-wave amplitudes will be removed. Thus, no nonsmoothness in the g^2 term needs to be dealt with. However, the input is now dependent on a particular choice of subtractions, that is to say, a particular field theory, rather than an "S-matrix theory" input, because any choice of polynomials in lowest order can be smoothed out and superficially made Regge-like by this procedure. We shall not discuss this kind of modification any further.

IV. DYNAMICAL METHODS

In this section we examine various common versions of the N/D method, to decide whether one does indeed obtain a Regge-like output amplitude when the input is chosen according to the above prescriptions.

Let us begin with the totally spinless case, scalar-scalar scattering. The input force is the Born term $B^j(s)$, which is of order g^2 and smooth in j with poles at the negative integers. The simplest procedure is the first-order, or "determinantal" method. The output amplitude is written

$$t^j(s) = B^j(s)/D(s), \tag{4.1}$$

where

$$D(s) = 1 - \frac{s-s_0}{\pi} \int_{s_t}^{\infty} \frac{B^j(s)ds'}{(s'-s)(s'-s_0)}. \tag{4.2}$$

Here, s_t is the physical threshold. The answer is evidently smooth in j. It also has the correct Regge trajectory behavior, which we may verify by looking at t^j near $j=-n$. Then $B^j(s)$ can be approximated by something of the form $b_n(s)/(j+n)$, and (4.1) becomes

$$t^j(s) \approx \beta(s)/[j-\alpha(s)], \tag{4.3}$$

provided we define

$$\begin{aligned} \beta(s) &= b_n(s), \\ \alpha(s) &= -n + \frac{s-s_0}{\pi} \int_{s_t}^{\infty} \frac{b_n(s')ds'}{(s'-s)(s'-s_0)}. \end{aligned} \tag{4.4}$$

Thus near each negative integer the first-order method is equivalent to keeping terms up to order g^2 in α and β.

In the "full" N/D method, one must choose an amplitude $\tilde{t}^j(s) = t^j(s)/\rho(s)$, so that unitarity becomes

$$\mathrm{Im}\tilde{t}^j(s) = \rho(s) |\tilde{t}^j(s)|^2 \tag{4.5}$$

for real j. The function $\rho(s)$ guarantees that $\tilde{t}^j(s)$ has no kinematical zeros or singularities. In general, it depends on j. If $\tilde{B}^j(s)$ is the input Born term to $\tilde{t}^j(s)$, one represents $\tilde{t}^j(s)$ by $N(s)/D(s)$ where

$$\begin{aligned} \mathrm{Im}N(s) &= \mathrm{Im}\tilde{B}^j(s) D(s), \\ \mathrm{Im}D(s) &= -\rho(s) N(s), \end{aligned} \tag{4.6}$$

on the left and right cuts, respectively.

In addition to these conditions on ImN and ImD, and some boundary condition for large s, one must require $\tilde{t}^j(s) \to \tilde{B}^j(s)$ as $g^2 \to 0$, which is a restriction on the subtractions. If the asymptotic conditions in s permit an unsubtracted integral in the N equation, the usual solution

$$\begin{aligned} N(s) &= \tilde{B}^j(s) + \frac{1}{\pi} \\ &\quad \times \int_{s_t}^{\infty} \frac{[\tilde{B}^j(s') - ((s-s_0)/(s'-s_0))\tilde{B}^j(s)]}{s'-s}, \\ D(s) &= 1 - \frac{s-s_0}{\pi} \int_{s_t}^{\infty} \frac{\rho(s')\tilde{B}^j(s')}{(s'-s)(s'-s_0)} ds', \end{aligned} \tag{4.7}$$

has the desired properties. Thus in the spinless case there is no difficulty obtaining a Regge-like solution once the input is properly chosen.

As soon as the external particles have spin, the Regge-like behavior of the various N/D solutions is not so unavoidable. Our example will again be the 2×2 scalar-vector amplitude. Since we are particularly interested in the behavior of $t^j(s)$ near $j=0$, we write the j-plane singularities there explicitly as in (3.5).

$$B^j(s) = \frac{1}{qW} \begin{pmatrix} b_1(s) & b_x(s)/Wj^{1/2} \\ b_x(s)/Wj^{1/2} & b_2(s)/sj \end{pmatrix}. \tag{4.8}$$

Recall from the preceding sections that b_x and b_2 are the g^2 terms in the corresponding elements of the residue functions for the α_0 trajectory.

The straightforward generalization of the first-order determinantal method to two channels is

$$t^j = B^j D^{-1} \equiv R/\det D, \tag{4.9}$$

which defines R. D is usually written

$$D = I - \frac{s-s_0}{\pi} \int_{s_t}^{\infty} \frac{B^j(s')ds'}{(s'-s)(s'-s_0)}. \tag{4.10}$$

Then the elements of the D matrix have the same singularities in j near $j=0$ as do the elements of B^j. We may remove them explicitly also, by writing

$$\begin{aligned} D_{11} &= 1 - d_{11}, \\ D_x &= D_{12} = D_{21} = -d_x/j^{1/2}, \\ D_{22} &= 1 - d_{22}/j. \end{aligned} \tag{4.11}$$

Then

$$\det D = D_{11}D_{22} - D_x^2 = \frac{D_{11}}{j}\left[j - d_{22} - \frac{d_x^2}{D_{11}} \right]. \tag{4.12}$$

Since $\det D \sim 1/j$ near $j=0$, we should expect R_{11} and R_{22} to be like $1/j$, while R_{12} and R_{21} will be like $1/j^{1/2}$. All the $R_{\mu\nu}$ pass the test except R_{21}:

$$R_{21}=B_{21}D_{22}-B_{22}D_{21}\sim 1/j^{3/2}. \tag{4.13}$$

Thus the sense and nonsense states do not decouple at $j=0$ in this frequently used approximation. This is a serious violation of reasonable behavior in many ways; in particular, the ${}^3P_0$ amplitude $t_{11}{}^0$ does not satisfy single-channel unitarity.

The trouble-making pole in j at $j=0$ in $t_{21}{}^j$ is absent in $t_{12}{}^j$, a fact which is possible because the method does not construct a symmetric output even from a symmetric input. In spinless many-channel applications, this well-known asymmetry has usually been accepted as just one more approximate feature of an already approximate method. When spins are present, it seems to be catastrophe.

Can we get an acceptable solution by using a symmetrized version of (4.8)? The simplest, and first suggested, is Bjorken's[16]:

$$t^j(s)=[B^{-1}D+D^TB^{-1}]^{-1}/2. \tag{4.14}$$

Equation (4.14) will suffice to symmetrize a spinless many-channel amplitude; but in the present problem we have only achieved both $t_{21}{}^j$ and $t_{12}{}^j\sim 1/j^{1/2}$ and so are worse off than before.

Several more complicated symmetrized versions have been suggested. Martin and Wali,[17] for example, used (4.9) but replaced (4.10) with

$$D=I-\frac{s-s_0}{2\pi}\int_{s_t}^{\infty}\frac{B(s')ds'}{(s'-s)(s'-s_0)}-B^{-1}(s)\frac{(s-s_0)}{2\pi}\times\int_{s_t}^{\infty}\frac{B(s')ds'}{(s'-s)(s'-s_0)}B(s). \tag{4.15}$$

Here, $t_{12}=t_{21}$ and both are now $\sim j^{1/2}$ at $j=0$, as desired; but one can also show that $t_{11}\sim j$, so that the sense-sense amplitude vanishes identically at $j=0$. The method is thus consistent with Regge-like behavior, but is clearly not an acceptable approximation near $j=0$, since there is no reason for a physical partial-wave amplitude to vanish identically.

An even more complicated symmetrized version has recently been proposed by Shaw,[18] which has many nice features in the many-channel spinless case. For the scalar-vector problem, however, Shaw's method has the same behavior near $j=0$ as Martin and Wali's. A somewhat different type of symmetrization is to write $t^j=B^{1/2}DB^{1/2}$, with D defined as in (4.10). This method would be very awkward to use in practice, since it involves square roots of matrices; fortunately, it does not work for the present problem either. We have not been able to find a symmetrized first-order version in which the sense-nonsense amplitudes decouple properly, and yet the physical $j=0$ amplitude does not vanish.

Apparently, then, symmetrization is not the solution to the difficulty displayed in (4.13). How can the coefficient of the bothersome $j^{-3/2}$ in R_{21} be made to vanish? This is possible if (4.10) or its symmetrized analog is replaced by a form in which the subtraction constant s_0 is different in each matrix element in such a way that the $j^{-3/2}$ parts of the two terms in (4.13) cancel. We shall see below how to do this naturally.

Next we come to the integral equation N/D method.[2] If enough general conditions are imposed on the solution, we shall discover that near $j=0$ it does have just the behavior described in Sec. III.

Let us repeat that we do not believe that a single-particle exchange N/D method with elastic unitarity is a particularly good quantitative approximation to anything. However, improvements such as the inclusion of higher order terms in the input forces, or the exchange of Regge trajectories, may share many of the same features.

The input force in our problem comes from the scalar exchange diagram, Fig. 1(b), and has been written down in (3.3). We shall find only the leading terms of the N/D solution near $j=0$, so that the B^j matrix will be taken in the form (3.5) with b_1, b_x, and b_2 independent of j. First we must find the diagonal matrix $\rho(s)$, with elements $\rho_{\mu\nu}=\rho_\mu\delta_{\mu\nu}$, which makes the function

$$\tilde{t}^j=\rho^{-1/2}t^j\rho^{-1/2}, \tag{4.16}$$

free of kinematical singularities and zeros, and then represent $\tilde{t}^j$ as ND^{-1}. In general, $\rho(s)$ is a function of j, but we shall need its form only at $j=0$. If $j\neq 0$, both helicity states have a nonzero projection onto ${}^3A_0$; so for correct threshold behavior, ρ must contain a factor q^{2j-1}, as well as the usual factor $1/W$. Thus at $j=0$, $\rho=1/qW$ times some diagonal matrix to take care of the singularities at $s=0$. By inspection of Eqs. (3.5), where we have anticipated the answer, and of the partial-wave projections of other scalar-vector diagrams satisfying the no-polynomials restriction (corresponding to the several terms in the appropriate Mandelstam representation) one convinces oneself that the correct functions are

$$\rho_1=1/qW;\quad \rho_2=1/qWs. \tag{4.17}$$

The Born approximation (input force) to $\tilde{t}^j$ near $j=0$ is just

$$\bar{B}^j=\begin{pmatrix} b_1 & b_x/j^{1/2} \\ b_x/j^{1/2} & b_2/j \end{pmatrix} \tag{4.18}$$

with

$$\begin{aligned} b_1(s)&=-\frac{g^2}{4\pi}\frac{q^2}{m^2}(E_s+xE_v)^2Q_0(x), \\ b_x(s)&=-\frac{g^2}{4\pi}\frac{q^2}{m}W(E_s+xE_v), \\ b_2(s)&=-(g^2/4\pi)q^2sx. \end{aligned} \tag{4.19}$$

[16] J. D. Bjorken (unpublished).
[17] A. W. Martin and K. C. Wali, Phys. Rev. **130**, 2455 (1963).
[18] G. Shaw, Phys. Rev. Letters **12**, 345 (1964).

Near $s=0$, the combination $(E_s+xE_v)^2$ is just s, so that b_x and b_2 have no singularities; they are first- and second-order polynomials in s, respectively. At large s, $(E_s+xE_v)^2\sim 1/s$, so as $s\sim\infty$,

$$\begin{aligned} b_1&\sim \ln s\,,\\ b_x&\sim s\,,\\ b_2&\sim s^2. \end{aligned} \tag{4.20}$$

Let $t=ND^{-1}\equiv R/\det D$. Unitarity requires

$$\mathrm{Im}D_{\mu\nu}=-\rho_\mu N_{\mu\nu} \tag{4.21}$$

on the unitarity cut; the condition on the left-hand cut is

$$\mathrm{Im}N_{\mu\nu}=(\mathrm{Im}\bar{B}_{\mu\nu'})D_{\nu'\nu}\,, \tag{4.22}$$

and all the elements of N and D are real-analytic.

We do not know the most general solution, but we can show how to construct, in a more or less natural way, a solution which agrees with the general description in Sec. III.

Let us require $D_{11}\to 1$ and $D_{22}\to 1$ as $g^2\to 0$; whereas D_{12} and D_{21} are to be of order g^2. Consequently,

$$\mathrm{Det}D=1+O(g^2)\,. \tag{4.23}$$

Then, if

$$N=\bar{B}+O(g^4)\,, \tag{4.24}$$

t^j will have the correct g^2 term when $j\neq 0$. Since $N_{\mu\nu}$ contains a term $\bar{B}_{\mu\nu}$, it is simple, albeit arbitrary, to take as boundary conditions

$$\begin{aligned} N_{11}&\sim \ln s\,,\\ N_{12},N_{21}&\sim s\,,\\ N_{22}&\sim s^2\,, \end{aligned} \tag{4.25}$$

just like (4.20). In a truly Regge-like theory, we could presumably anticipate a less singular asymptotic behavior than this. Define

$$I=\frac{1}{\pi}\int_{s_t}^{\infty}\frac{\rho_2(s')ds'}{s'-s}\,. \tag{4.26}$$

Since $\mathrm{Im}\bar{B}_{22}=\mathrm{Im}\bar{B}_{21}=\mathrm{Im}\bar{B}_{12}=0$, the equations (4.21) and (4.22) decouple for each pair of indices, and the problem may be done separately for each μ and ν.

A simple choice for $N_{2\nu}$ satisfying (4.24) is just

$$N_{22}=b_2/j\,, \tag{4.27}$$

$$N_{21}=b_x/j^{1/2}. \tag{4.28}$$

In general, any polynomials could be added to these expressions, without violating the basic equations (4.21) and (4.22). Some particular choice of such polynomials is equivalent to the commonly written formula

$$N(s)=B(s)+\frac{1}{\pi}\int\frac{ds'}{s'-s}\left[B(s')-\frac{s-s_0}{s'-s_0}B(s)\right]\rho(s')N(s')\,,$$

since the second term is a polynomial when $\mu=2$. However, because this equation does not permit the decoupling of the four N's, it is a rather inconvenient form to use.

The amount of ambiguity in choice of subtractions and additive polynomials has always been a source of embarrassment in the N/D method when the input amplitude B does not vanish at large s. We are merely offering *a* prescription for handling these ambiguities consistent with the restrictions imposed by smoothness in j.

The most general forms for $D_{2\nu}$ are

$$D_{22}=P_{22}-(b_2/j)I\,, \tag{4.29}$$

$$D_{21}=P_{21}-(b_x/j^{1/2})I\,, \tag{4.30}$$

where P_{21} and P_{22} are arbitrary polynomials. We take $P_{22}=1$, $P_{21}=0$, to agree with the $g^2\to 0$ limit. More generally, any polynomial of order g^4 or higher could be added. It will become clear below that this particular choice is sufficient to make the coefficient of $j^{-3/2}$ in R_{21} vanish.

Next we discuss the $\mu=1$ elements. N and D may each be multiplied by an arbitrary nonsingular constant matrix, and still satisfy all the conditions.

We therefore normalize N_{12} and D_{12} in a convenient way:

$$D_{12}=-\frac{s-s_0}{\pi}\int_{s_t}^{\infty}\frac{\rho_1(s')N_{12}(s')ds'}{(s'-s)(s'-s_0)}\,, \tag{4.31}$$

$$D_{11}=1-\frac{1}{\pi}\int_{s_t}^{\infty}\frac{\rho_1(s')N_{11}(s')ds'}{s'-s}\,. \tag{4.32}$$

D_{11} is normalized to 1 at $s\sim\infty$, and D_{12} to $\ln s$ times some function of s_0; s_0 is not a normalization point, for we determine it below. N_{12} has the form

$$N_{12}=b_1D_{12}+\frac{s-s_1}{\pi}\int_{s_t}^{\infty}\frac{b_1(s')\rho_1(s')N_{12}(s')ds'}{(s'-s)(s'-s_1)}+\frac{b_x(s)}{j^{1/2}}+\frac{a}{j^{1/2}}\,. \tag{4.33}$$

The first two terms satisfy (4.21) and (4.22). Since both are of order g^4, $b_x/j^{1/2}$ is added explicitly to agree with (4.24). Any linear polynomial $(a+bs)/j^{1/2}$ may still be added, provided a and b are both of order g^4. We have normalized N_{12} at $s\sim\infty$ by setting $b=0$. Finally,

$$N_{11}=b_1D_{11}+\frac{1}{\pi}\int_{s_t}^{\infty}\frac{b_1(s')\rho_1(s')N_{11}(s')ds'}{s'-s}\,. \tag{4.34}$$

In each case we have made the smallest number of subtractions possible, consistent with (4.25). The solution depends on three numbers a, s_0, and s_1.

Observe that, because we did not include polynomials of order g^4 in P_{22} and P_{21},

$$R_{21}=N_{21}D_{22}-N_{22}D_{21}=b_x/j^{1/2}. \tag{4.35}$$

This is consistent with the condition that sense and nonsense decouple at $j=0$.

Now symmetry requires $R_{21}=R_{12}$, or

$$R_{12}=N_{12}D_{11}-N_{11}D_{12}=b_x/j^{1/2}. \tag{4.36}$$

The combination $N_{12}D_{11}-N_{11}D_{12}$ is evidently a linear polynomial with the correct coefficient of s; and so will agree with $b_x/j^{1/2}$ provided it vanishes at the zero of b_x; this provides one condition on a, s_0, and s_1. R_{11} and R_{22} also have the correct Born terms, so

$$\bar{t}^j=\bar{B}+O(g^4) \tag{4.37}$$

when $g\neq 0$. Let us examine R_{11} more closely.

$$\begin{aligned}\bar{t}_{11}{}^j&=\frac{R_{11}}{\det D}=\frac{N_{11}D_{22}-N_{12}D_{21}}{D_{11}D_{22}-D_{12}D_{21}}\\&=\frac{N_{11}(j-b_2I)+(j^{1/2}N_{12})b_xI}{D_{11}(j-b_2I)+(j^{1/2}D_{12})b_xI}.\end{aligned} \tag{4.38}$$

Now we can continue to $j=0$ to find the ${}^3P_0$ amplitude:

$$\bar{t}_{11}{}^0=\frac{(j^{1/2}N_{12})b_x-N_{11}b_2}{(j^{1/2}D_{12})b_x-D_{11}b_2}. \tag{4.39}$$

The quantities $j^{1/2}N_{12}$ and $j^{1/2}D_{12}$ are in fact independent of j. Since $t_{11}{}^0$ is a pure P-wave amplitude,

$$\bar{t}_{11}{}^0(s_t)=(d/ds)\bar{t}_{11}{}^0(s_t)=0, \tag{4.40}$$

which are the two remaining conditions on s_0, s_1, and a. As $g^2\to 0$,

$$\bar{t}_{11}{}^0\to(b_1b_2-b_x{}^2)/b_2 \tag{4.41}$$

in agreement with our expectations in previous sections.

The g^2 term in $\bar{t}_{11}{}^0$ is independent of the parameters a, s_0, s_1; but we have already remarked that it has the correct threshold behavior. For, suppose instead of helicity states we had chosen eigenstates of orbital angular momentum l as basis vectors. The $l=j+1$ state is sense, the $l=j-1$ state is nonsense, at $j=0$. The elements of $\bar{B}^j$ would have been new functions $b_1'(s)$, $b_x'(s)/j^{1/2}$, and $b_j'(s)/j$ where, near threshold, $b_x'\sim q^2$ and $b_2'\sim q^4$. Therefore,

$$b_1b_2-b_x{}^2=b_1'b_2'-(b_x')^2\sim q^4 \tag{4.42}$$

near $s=s_t$, so (4.41) has the correct P-wave behavior to lowest order in g^2 as well.

To see the Regge behavior of our solution, write

$$\det D=D_{11}(j-\alpha_0)/j, \tag{4.43}$$

defining the trajectory α_0 by

$$\alpha_0=\left(b_2-\frac{(j^{1/2}D_{12})b_x}{D_{11}}\right)I. \tag{4.44}$$

We can easily find the residues from the definition (2.11):

$$\bar{t}_x{}^j=\frac{R_{12}}{\det D}=\frac{R_{21}}{\det D}=\frac{j^{1/2}\beta_x}{j-\alpha_0} \tag{4.45}$$

gives

$$\beta_x=b_x/D_{11}. \tag{4.46}$$

Similarly,

$$R_{22}=N_{22}D_{11}-N_{21}D_{12}=D_{11}\alpha_0/jI. \tag{4.47}$$

So,

$$\bar{t}_{22}=\beta_2/(j-\alpha_0), \tag{4.48}$$

provided

$$\beta_2=\alpha_0/I. \tag{4.49}$$

Finally,

$$jR_{11}=N_{11}(j-b_2I)+(j^{1/2}N_{12})b_xI. \tag{4.50}$$

To find the residue, replace j by α_0. Then, since

$$\alpha_0-b_2I=-(j^{1/2}D_{12})b_xI/D_{11}, \tag{4.51}$$

$$\begin{aligned}jR_{11}=I[&(j^{1/2}N_{12})b_x\\&-(j^{1/2}D_{12})b_xN_{11}/D_{11}]+N_{11}(j-\alpha).\end{aligned} \tag{4.52}$$

Therefore,

$$\bar{t}_{11}=\frac{R_{11}}{\det D}=\frac{\beta_1}{j-\alpha}+\frac{N_{11}}{D_{11}}. \tag{4.53}$$

From the symmetry condition (4.36),

$$(j^{1/2}N_{12})=[b_x+N_{11}(j^{1/2}D_{12})]/D_{11}. \tag{4.54}$$

Substitute (4.54) into the first term of (4.52):

$$\begin{aligned}\beta_1&=(I/D_{11})[(j^{1/2}N_{12})b_x-(j^{1/2}D_{12})b_x(N_{11}/D_{11})]\\&=Ib_x{}^2/D_{11}{}^2=I\beta_x{}^2=\beta_x{}^2\alpha_0/\beta_2.\end{aligned} \tag{4.55}$$

The residues factor as they obviously must. In the present approximation, then,

$$\bar{t}_{11}{}^j=\frac{1}{j-\alpha_0}\begin{bmatrix}\beta_x{}^2\alpha_0/\beta_2 & j^{1/2}\beta_x\\ j^{1/2}\beta_x & \beta_2\end{bmatrix}+\begin{bmatrix}N_{11}/D_{11} & 0\\ 0 & 0\end{bmatrix}. \tag{4.56}$$

Notice that $\mathrm{Im}\alpha_0=\rho_2\beta_2$, in agreement with (2.14). In fact, α_0 and β_2 vanish at the same places; in the next section we shall show this to be a general property of α_0. Because of (4.49), α_0 is a sense-choosing trajectory. (Incidentally, the zeros in the second matrix are simply due to our having omitted from the input forces terms with poles at $j=-1, -2, -3, \cdots$, etc.)

The second term in (4.56) represents, in our approximation, the contribution of all the usual trajectories which go to negative integers as $g^2\to 0$. The new trajectory α_0 appears explicitly, since we have kept only the leading terms in j near $j=0$. This solution can have a ${}^3P_0$ bound state in either of two ways. If a spin-zero particle appears when $D_{11}=0$ from the second term in (4.56), it lies on a trajectory which has been brought from some negative integer $j=-n$ up to $j=0$, and is produced dynamically in a way completely analogous to bound states in spinless problems. But a particle also appears when $\alpha_0=0$, in which case it lies on the new trajectory. Since I does not vanish, the condition is, from (4.44),

$$b_2=(j^{1/2}D_{12})b_x/D_{11} \tag{4.57}$$

or, in lowest order, just

$$b_2=0. \tag{4.58}$$

This bound state has a behavior quite different from usual ones; the next section is devoted to studying it.

The ${}^3P_0$ amplitude $\hat{t}_{11}{}^0$ can be written in a one-channel N/D form. What is the equivalent single-channel problem? It is sufficient to require that the solution look like (4.39), and that

$$\mathrm{Im}N(s)=\mathrm{Im}b_1(s)D(s), \tag{4.59}$$

$$\mathrm{Im}D(s)=-\rho_1(s)N(s), \tag{4.60}$$

in the proper regions. Since, as $g^2\to 0$,

$$N\to b_1b_2-b_x{}^2, \tag{4.61}$$

$$D\to b_2, \tag{4.62}$$

any N/D solution in which the 22 and crossed amplitudes are not taken into account in this way cannot agree with Regge-like behavior. A general form for the solution is

$$N=b_1D+\frac{P_1(s)}{\pi}\int_{s_t}^{\infty}\frac{\rho_1(s')b_1(s')N_1(s')ds'}{(s'-s)P_1(s')}+P_2, \tag{4.63}$$

$$D=P_3-\frac{P_4(s)}{\pi}\int_{s_t}^{\infty}\frac{\rho_1(s')N(s')ds'}{(s'-s)P_4(s')}. \tag{4.64}$$

The four functions $P_i(s)$ are polynomials such that as $g^2\to 0$,

$$\begin{aligned}P_3(s)&\to b_2(s),\\ P_2(s)&\to -b_x{}^2(s).\end{aligned} \tag{4.65}$$

A particular choice, of course, is necessary to agree with our particular solution (4.39).

We can now find a "correct" analog of the first-order method, for we have shown how to choose the subtractions in D_{22} and D_{21} so that the coefficient of $j^{-3/2}$ in R_{21} vanishes. For example, one may simply write

$$\begin{aligned}&N=\tilde{B},\\ &D_{11}=1-\frac{1}{\pi}\int_{s_t}^{\infty}\frac{\rho_1(s')b_1(s')}{s'-s}\equiv 1-d_1,\\ &D_{12}=-\frac{s-s_0}{\pi j^{1/2}}\int_{s_t}^{\infty}\frac{\rho_1(s')b_x(s')ds'}{(s'-s)(s'-s_0)}\equiv\frac{-d_{12}}{j^{1/2}},\\ &D_{22}=1-b_2I/j,\\ &D_{21}=-b_xI/j^{1/2}.\end{aligned} \tag{4.66}$$

The threshold behavior at $j=0$ is automatic, for

$$\hat{t}_{11}{}^0=\frac{b_1-b_x{}^2/b_2}{D_{11}-d_{12}b_x/b_2}. \tag{4.67}$$

Now there is no condition on s_0, so this parameter is the usual arbitrary subtraction which occurs in this approximation. Since

$$R_{12}\neq b_x/j^{1/2},$$

$\hat{t}^j$ is still not symmetric except at $j=0$. In fact, R_{12} is no longer a polynomial, so that no choice of s_0 can guarantee symmetry. Now we have a first-order solution which can be obtained without solving any integral equations:

$$\hat{t}=\frac{1}{l-\alpha_0}\begin{bmatrix}\beta_1 & j^{1/2}\beta_x\\ j^{1/2}\beta_x & \beta_2\end{bmatrix}+\begin{bmatrix}b_1/(1-d_1) & 0\\ 0 & 0\end{bmatrix}, \tag{4.68}$$

with

$$\alpha_0=b_2I+b_xd_{12}I/(1-d_1), \tag{4.69}$$

$$\begin{aligned}&\beta_2=\alpha_0/I,\\ &\beta_{21}=b_x/(1-d_1),\\ &\beta_{12}=b_x+d_{12}b_1/(1-d_1),\\ &\beta_1=[b_1(\alpha_0-b_2I)-b_xd_{12}]/(1-d_1).\end{aligned} \tag{4.70}$$

This type of solution is similar in structure to (4.56), but is not symmetric and does not satisfy (4.22).

Notice that in order to obtain a sensible generalization of the first-order determinantal method to a spinning problem, it was necessary to choose the subtractions in the $D_{\mu\nu}$ at different points. In particular, $D_{12}\neq D_{21}$. However, by a suitable choice of s_0 (namely, $s_0=$ the zero of b_x), the third of Eqs. (4.66) can be replaced by

$$d_{12}=-j^{1/2}D_{12}=b_x(s)I'(s), \tag{4.71}$$

where

$$I'(s)=\frac{1}{\pi}\int_{s_t}^{\infty}\frac{\rho_1(s')ds'}{s'-s}. \tag{4.72}$$

Then if ρ_1 and ρ_2 were equal, we would have $I'=I$ and $D_{12}=D_{21}$.

Instead of expanding N and D in powers of g^2, we can equally well expand R and $\det D$. This leads to a result equivalent to the method of asymptotic unitarity.[7,19] We obtain

$$\hat{t}=\frac{1}{j-\alpha_0}\begin{pmatrix}b_x{}^2\alpha_0/b_2 & j^{1/2}b_x\\ j^{1/2}b_x & b_2\end{pmatrix}+\begin{pmatrix}b_1 & 0\\ 0 & 0\end{pmatrix}, \tag{4.73}$$

with $\alpha_0=b_2I$. It may be remarked that this result is at least as natural a generalization of the one-channel determinantal method as Eqs. (4.68) to (4.70). It has the desirable features of symmetry, decoupling of sense and nonsense at $j=0$, and the correct threshold behavior in the sense-sense amplitude. Furthermore, there are no free parameters analogous to the s_0 of the usual generalization of the determinantal method. It is, of course, not unitary to all orders in g^2. This approximation serves as the basis for the "spinological bootstrap" discussed in the next section.

[19] M. Gell-Mann, M. L. Goldberger, F. E. Low, V. Singh, and F. Zachariasen, Phys. Rev. 133, B161 (1964).

We conclude our survey of common techniques by remarking briefly on pole approximations to the N/D equations in the two-channel case. These have frequently been used for qualitative arguments, as they have the great advantage that the resulting N/D equations can be solved exactly.[20] Suppose, for example, the input is chosen so that near $j=0$,

$$\bar{B}(s)=\frac{1}{s-s_p}\begin{pmatrix} b_1 & b_x/j^{1/2} \\ b_x/j^{1/2} & b_2/j \end{pmatrix}, \tag{4.74}$$

where b_1, b_2, and b_x are constants. The solution is simple. There is a Regge trajectory going to 0:

$$\alpha_0=\left[b_2+\frac{b_x{}^2}{1-b_1d_1}\right]d_2, \tag{4.75}$$

where

$$d_i=\frac{s-s_p}{\pi}\int_{s_t}^{\infty}\frac{\rho_i(s')ds'}{(s'-s)(s'-s_p)^2}.$$

This is completely analogous to our previous solution, even though here $\bar{B}_x$ and $\bar{B}_2$ are not polynomials. The residue functions $\beta_{\mu\nu}$ all contain the factor $1/s-s_p$. In a two-pole approximation, the input might be

$$\bar{B}(s)=\frac{1}{s-s_a}\begin{pmatrix} a_1 & a_x/j^{1/2} \\ a_x/j^{1/2} & a_2/j \end{pmatrix} + \frac{1}{s-s_b}\begin{pmatrix} b_1 & b_x/j^{1/2} \\ b_x/j^{1/2} & b_2/j \end{pmatrix}. \tag{4.76}$$

One then obtains *two* Regge trajectories going to $j=0$ as a_i, $b_i\to 0$, with residues proportional to $1/s-s_a$ and $1/s-s_b$, respectively. The solution to an N-pole approximation probably has N such trajectories, which is a bit disturbing. On the other hand, since this method approximates polynomials by poles, one should perhaps not expect too much of it.

V. SPINOLOGICAL DYNAMICS

In this final section we study some novel features of dynamical calculations with the new trajectory α_0. An "ordinary" trajectory α goes to some negative integer $-n$ in the $g^2\to 0$ limit, and is determined by an expression of the form

$$\alpha(s)=-n+\frac{g^2}{\pi}\int_{s_t}^{\infty}\frac{f(s')ds'}{(s'-s)}+O(g^4), \tag{5.1}$$

possibly with some subtractions. A physical spin-0 bound state or resonance then lies on this trajectory at an energy $s^{1/2}$ whenever

$$\alpha(s)=0. \tag{5.2}$$

Because of the $-n$ term in (5.1), Eq. (5.2) is impossible if g^2 is sufficiently small; "dynamics" or "forces" are needed to bring the trajectory up to a physical j. The energy of the bound state depends on g^2 in an orthodox way, increasing as the coupling strength is made smaller.

As we observed in Sec. IV, the trajectory α_0, in contrast, starts at a physical angular momentum $j=0$. No dynamics at all seems necessary to produce a bound state lying on this trajectory! The energy s at which $\alpha_0(s)=0$ depends essentially on spinological factors alone. This is the circumstance which made possible the conjecture[21] that in spinor-vector scattering (electrodynamics with massive photons) the spinor can, in the Regge sense, be like a dynamical bound state even if the coupling is weak.

It should be emphasized, of course, that in a truly dynamical theory, the weak coupling limit cannot really be defined, since the solution is supposed to exist for only a single value of g^2. In such a case there is nothing in the behavior of the trajectory α_0 to conflict with one's intuition.

Let us look more closely at the trajectory α_0 in our scalar-vector example. It contributes to the partial-wave amplitude a term like

$$\frac{1}{j-\alpha_0}\begin{bmatrix} \beta_x{}^2\alpha_0/\beta_2 & j^{1/2}\beta_x \\ j^{1/2}\beta_x & \beta_2 \end{bmatrix}. \tag{5.3}$$

The g^2 terms of β_x and β_2 can be computed from the Born approximation.

Does this trajectory choose sense or nonsense[7,11] when $\alpha_0=0$? Unless either α_0 vanishes at the zeros of β_2, or β_x vanishes at these points, the sense-sense element $\beta_x{}^2\alpha_0/\beta_2$ has poles at these zeros for all values of j, producing a bound state at some energy for all angular momenta, which is as intolerable in a Regge-like theory as a fixed j-plane pole. In the scalar-vector and spinor-vector theories, at least, except perhaps accidentally for certain discrete values of the masses and coupling constants, β_x is not zero at the zeros of β_2. Therefore, $\alpha=0$ at these points. Then there is no pole in the nonsense-nonsense amplitude, and the trajectory chooses sense.[22]

Notice from Eq. (4.49) that in the particular N/D approximation we used in the previous section, the zeros of β_2 and α were indeed in the same place. In dispersion-theory language, there is a pole in the physical 0^+ partial-wave amplitude at an energy-squared s_p such that $\beta_2(s_p)=0$; the pole has a residue

$$-\beta_x{}^2(s_p)\Big/\frac{d}{ds}\beta_2(s_p). \tag{5.4}$$

[20] See, for instance, L. A. P. Balázs, Phys. Rev. **128**, 1939 (1962).

[21] M. Gell-Mann and F. Zachariasen, Phys. Letters (to be published).

[22] The triplet nucleon-nucleon amplitude with parity $(-1)^j$ is also a 2×2 matrix with one sense channel and one nonsense channel at $j=0$. If the force is provided by pion exchange, and the amplitude is expanded in the pion-nucleon coupling constant, this theory appears to choose the other alternative, namely $\beta_x=0$ whenever $\beta_2=0$. In this case, α_0 is a nonsense-choosing trajectory if it goes through zero!

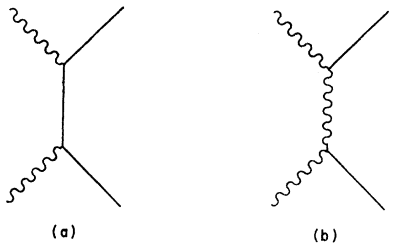

FIG. 2. PS and V poles in the PS$-V$ amplitude.

Of course, we do not know β_2 exactly, and therefore cannot prove that it must have zeros at all. All that is known is that in our examples it does have zeros to order g^2, since $b_2(s)$ is a polynomial, so that $\beta_2(s)$ certainly has zeros if the force is weak enough. Notice how unusual the positions of these poles in the amplitude are. As $g^2 \to 0$, they approach fixed, finite, values.

If the vector particle in our theory is neutral, the new 0^+ bound state has the same quantum numbers as the external scalar particle. If we identify them, we obtain what might be called a spinological bootstrap:

$$\beta_2(\mu^2)=0\,,$$
$$\beta_x^2(\mu^2)\Big/\frac{d}{ds}\beta_2(\mu^2)=-\frac{2g^2}{4\pi}q^4(\mu^2)\frac{\mu^2}{m^2}\,. \tag{5.5}$$

These equations replace the usual mass and residue conditions.

One may compute β_x and β_2, for example, by any of the approximate methods described in Sec. IV. First-order perturbation theory is probably as good as anything else. In the spinless case, we saw that near a negative integer $-n$ and near a pole in s, this method was equivalent to retaining terms up to order g^2 in the appropriate α and β. The analogous thing here would be to do the same thing for α_0, and write

$$\tilde{t}_{11}{}^0(s)\approx -b_x^2(s)/b_2(s) \tag{5.6}$$

near a zero in b_2. This is what one obtains by using the asymptotic unitarity approximation of Eq. (4.73). We might do better by including higher order corrections to the numerator and denominator as in (4.34) or (4.50). But we are only describing a model anyway, and (5.6) is the simplest version which is capable of producing a bound state; it will serve to illustrate how a spinological bootstrap works.

The bootstrap equations are then

$$b_2(\mu^2)=0\,,$$
$$b_x^2(\mu^2)\Big/\frac{d}{ds}b_2(\mu^2)=2\frac{g^2}{4\pi}q^4(\mu^2)\frac{\mu^2}{m^2}\,. \tag{5.7}$$

These are algebraic, polynomial equations, whereas the usual bootstrap calculations involve integrals and cannot be solved explicitly. In some respects it appears that the *absence* of spin is a complication!

Let us try the mass condition first. From the third of Eqs. (4.19),

$$b_2(s)=-\frac{g^2}{4\pi}q^2sx=-\frac{g^2}{16\pi}[(m^2-\mu^2)^2+2m^2s-s^2]\,, \tag{5.8}$$

$$b_2(\mu^2)=-\frac{g^2}{4\pi}\frac{m^4}{4}\,. \tag{5.9}$$

There can be a solution only if $m^2=0$. In that case,

$$b_2(s)=-\tfrac{1}{4}(\mu^2-s)(\mu^2+s)\,. \tag{5.10}$$

$b_2(s)$ then does vanish at $s=\mu^2$, but also at $s=-\mu^2$; unless $\mu^2=0$ also, there will be a ghost at $s=-\mu^2$.

Thus this method applied to the scalar-vector problem fails to give a sensible answer. Perhaps a reasonable solution could be obtained by improving our simple approximation (5.7); we have no way of knowing. In any case, the real world can in no approximation be considered to contain one scalar and one vector particle; we should probably be encouraged rather than dismayed.[23]

Can this method be a bootstrap theory of π and ρ mesons? Except for isotopic spin factors, the same amplitudes describe $\pi-\rho$ scattering with π exchange providing the force, for we have used the relative parity of the two particles only to give names to the states. Thus, we can let the 2×2 matrix t^j be the parity $=-(-1)^j$, isotopic spin one, $\pi-\rho$ amplitude, and bootstrap the 0^- channel. The functions b_1, b_x, and b_2 remain the same except for the charge-space crossing matrix.

For $\pi\rho$ scattering in the $I=1$ state, the relevant crossing matrix element for Fig. 1(b) is just 1. Therefore, we get the same equations as before, and again there is no solution.

It would be nice to find some physical situation for which Eqs. (5.5) are meaningful, in order to demonstrate that there is not something inherently absurd in them. Let us try to approximate the real world even better by promoting the π and ρ to octets of PS (pseudoscalar) mesons of mass μ and V (vector) mesons of mass m, coupled in an SU(3) symmetric way. Figure 1(b) now represents the PS octet exchange force, and is proportional to the PS–PS–V coupling constant g^2. In order to avoid the same trivial solution as before, we add the V exchange diagram, Fig. 1(c), to the input PS$-V$ force. Let h/μ be the usual PS$-V-V$ coupling constant; Fig. 1(c) is then proportional to the dimensionless constant h^2. The g^2 coupling is pure f, while the h^2 coupling is pure d.

In order to bootstrap the PS octet, we need our matrix $t^j(s)$ for parity $-(-1)^j$ projected into the antisymmetric octet combination of PS$+V$ states. To do this, we compute the pole diagrams, Fig. 2, as if the

[23] The analogous computation for spinor-vector scattering is, in contrast, an identity (see Ref. 21).

mesons had no charges or strangeness, which defines the magnitudes of g^2 and h^2; then compute the projections of Figs. 1(b) and 1(c) in the same way, and multiply each term by the appropriate element of the SU(3) crossing matrix. Thus we can obtain the input to first order in g^2 and h^2.

The PS pole, Fig. 2(a), in the antisymmetric octet $t_{11}{}^0$ amplitude, normalized like $b_1(s)$ in (3.5), is then

$$t_{11}{}^0=-\frac{2g^2}{4\pi}\frac{q^4\mu^2}{m^2}\frac{1}{s-\mu^2}. \tag{5.11}$$

Since $g^2=\frac{3}{2}g_{\pi\pi\rho}{}^2$, the experimental value of $g^2/4\pi$ from ρ decay is about 3. The vector octet pole in the symmetric 1^- amplitude, normalized to $(q/W)e^{i\delta}\sin\delta$, is

$$-\frac{h^2}{4\pi}\frac{m^2}{3\mu^2}\frac{1}{s-m^2}. \tag{5.12}$$

If the ω were a pure octet state, we would have $h^2=(5/3)\times h_{\pi\rho\omega}{}^2$, and would guess $h^2/4\pi\approx 0.5$ from ω decay. But this is probably incorrect.

The pseudoscalar exchange force matrix elements are just the functions (3.3) and (3.5) multiplied by the crossing matrix element $\frac{1}{2}$. The h^2 input terms can be computed in the same way, namely by projecting out the partial wave amplitudes from Fig. 1(c) for high j and continuing down in j. We calculate these functions as if the particles were chargeless, and then multiply by the crossing matrix element $-\frac{1}{2}$.

Let $B_V{}^j$ be the contributions of Fig. 1(c), normalized like (3.3) to $e^{i\delta}\sin\delta$. In the same basis as before,

$$\begin{aligned}
B_{V,11}{}^j&=-\frac{1}{2}\frac{h^2}{4\pi}\frac{q}{W}\frac{m^2}{4\mu^2}(1-x_v{}^2)Q_j(x_v),\\
B_{V,x}{}^j&=+\frac{h^2}{4\pi}\frac{1}{2}\frac{qm}{4\mu^2W}\left[\frac{1}{j+1}\right]^{1/2}\\
&\quad\times(E_s+x_vE_v)[Q_{j-1}(x_v)-x_vQ_j(x_v)],\\
B_{V,22}{}^j&=+\frac{h^2}{4\pi}\frac{1}{2}\frac{q}{4\mu^2W}\frac{1}{j+1}\\
&\quad\times\{(q^2+2E_sE_v+x_vE_v{}^2)Q_{j-1}(x_v)\\
&\quad+[E_s{}^2-q^2x_v+j(E_s+x_vE_v{}^2)]Q_j(x_v)\},
\end{aligned} \tag{5.13}$$

where

$$x_v=1+\frac{m^2+2\mu^2-s}{2q^2}. \tag{5.14}$$

Of course, the new Born term has the same structure near $j=0$ as the old one. Factoring out the kinematical singularities near $j=0$ as in (3.5), we obtain

$$\begin{aligned}
b_{V,1}&=-\frac{1}{2}\frac{h^2}{4\pi}\frac{m^2}{4\mu^2}q^2(1-x_v)^2Q_0(x_v),\\
b_{V,x}&=-\frac{1}{2}\frac{h^2}{4\pi}\frac{m}{4\mu^2}q^2(E_s+x_vE_v)W,\\
b_{V,2}&=-\frac{1}{2}\frac{h^2}{4\pi}\frac{q^2s}{4\mu^2}(q^2+2E_sE_v+x_vE_v{}^2).
\end{aligned} \tag{5.15}$$

It is easy to check that b_x and b_2 are still polynomials. As usual, vector exchange is more divergent than scalar exchange: b_2 is now a cubic in s.

Now we repeat the calculation with b_2 and b_x taken as the sum of the two contributions. We may write b_2 as a polynomial in s:

$$b_2(s)=-\frac{1}{2}\frac{g^2}{4\pi}\frac{G(s)}{4}+\frac{1}{2}\frac{h^2}{4\pi}\frac{1}{32\mu^2}H(s), \tag{5.16}$$

where

$$\begin{aligned}
G(s)&=-s^2+2m^2s+(m^2-\mu^2)^2,\\
H(s)&=s^3-(2\mu^2+5m^2)s^2+(3m^4+4m^2\mu^2+\mu^4)s\\
&\quad+m^2(m^2-\mu^2)^2.
\end{aligned} \tag{5.17}$$

The mass condition becomes

$$0=b_2(\mu^2)=+\frac{m^4}{8}\left[-\frac{g^2}{4\pi}+\frac{h^2}{4\pi}\frac{(m^2+\mu^2)}{8\mu^2}\right]. \tag{5.18}$$

Now there may be a solution even if neither particle is massless. From (5.18), if $m\neq 0$,

$$\mu^2/m^2=h^2/(8g^2-h^2). \tag{5.19}$$

A solution exists with real masses and coupling constants as long as the residue condition can be satisfied with $h^2<8g^2$. The derivative of $b_2(s)$ follows from (5.16) and (5.17):

$$\frac{d}{ds}b_2(\mu^2)=-\frac{g^2}{4\pi}\frac{(m^2-\mu^2)}{4}+\frac{3}{64}\frac{h^2}{4\pi}\frac{m^2}{\mu^2}(m^2-2\mu^2). \tag{5.20}$$

Similarly,

$$\begin{aligned}
b_x{}^2(\mu^2)=\frac{m^2}{16\mu^2}\Bigg\{&\left(\frac{g^2}{4\pi}\right)^2\frac{\mu^2}{4m^2}(3m^2-4\mu^2)^2\\
&+\left(\frac{h^2}{4\pi}\right)^2\frac{m^2}{64\mu^2}(2m^2-3\mu^2)^2\\
&-\frac{g^2}{4\pi}\frac{h^2}{4\pi}\frac{1}{8}(6m^4-17m^2\mu^2+12\mu^4)\Bigg\}.
\end{aligned} \tag{5.21}$$

The residue condition is a polynomial equation of the form

$$A_{g^2}\left(\frac{g^2}{4\pi}\right)^2+A_{gh}\left(\frac{g^2}{4\pi}\right)\left(\frac{h^2}{4\pi}\right)+A_{h^2}\left(\frac{h^2}{4\pi}\right)^2=0. \tag{5.22}$$

Assuming that none of μ, m, g, and h are zero, set

$$\begin{aligned}
y&=h^2/g^2,\\
z&=m^2/\mu^2.
\end{aligned} \tag{5.23}$$

Then (5.22) becomes

$$\left(-4+6z-\frac{9}{4}z^2+\frac{1}{2}z^3\right)+\left(\frac{3}{2}z-\frac{7}{8}z^2+\frac{3}{16}z^3-\frac{3}{32}z^4\right)y+\left(\frac{9}{64}z^2-\frac{3}{16}z^3+\frac{1}{16}z^4\right)y^2=0. \quad (5.24)$$

This equation is to be solved together with

$$z=(8-y)/y \quad (5.25)$$

from (5.20). Combining (5.24) and (5.25), we get a fifth-order algebraic equation for y:

$$-128+304y-256y^2+\frac{359}{4}y^3-\frac{365}{32}y^4+\frac{25}{64}y^5=0. \quad (5.26)$$

There are 3 solutions in the permitted range:

$$\begin{aligned} &y=1.08, \quad y=1.6, \quad y=6.26, \\ &z=6.4, \quad z=4.0, \quad z=0.27. \end{aligned} \quad (5.27)$$

This, then, is the analog of the first-order, or determinantal, bootstrap calculation of a PS octet from PS$-V$ scattering in an SU(3) symmetric model, under the assumption that the PS octet lies on the leading trajectory. It is amusing that the analogous problem has no solution for SU(1) or SU(2).

To be fair, we must point out that because $b_2(s)$ is a cubic polynomial in s, $t_{11}{}^0$ will have two other poles in addition to the self-consistent one found here. If either of our solutions is to be the first-order approximation of a physical situation, these "ghosts" must disappear when higher orders are included correctly. Such "ghosts" possibly occur in many ordinary first-order calculations but are not noted since the solution is not known in a closed analytic form like (5.16).

This particular ugliness is absent in the analogous spinor-vector calculation.[21] There, $b_2(s)$ is linear, so only the self-consistent pole appears.

Our earlier remarks on bootstrap calculations in general will serve as our comment on the physical meaning of the numerical results. At least by displaying an example with (unfortunately 3) acceptable solutions, we have shown that there is no obvious inconsistency or absurdity.

Nevertheless it is important to remember the origin of the solutions we have obtained. The states exist because of the vanishing of $b_2(s)$, the input force from the nonsense channel. This nonsense force influences the physics of the sense channel at $j=0$ only because it is singular there. The singularity exists because the input was calculated only from the single exchange diagram, in which all particles are treated as "elementary." In fact, the external particles should have structure, although this will not show up in the input force without the inclusion of more complicated diagrams.

Hence it is likely that the approximation of computing $\beta_2(s)$, and hence the zeros of $\alpha_0(s)$, from an N/D method in which the structure is neglected is a very poor one, and that little importance should be attached to the predictions of such a model. If higher order terms had been included in the input, the result might be very different, at least quantitatively. One should certainly therefore view the practical use of the N/D method in these problems with suspicion. Indeed, the result we have obtained might be viewed as showing the shortcomings of the N/D method with simple exchange inputs rather than as an approximation to nature.

We conclude by summarizing our principal points. We have described some novel features of the Regge-trajectory structure of scattering amplitudes for spinning particles; especially those features associated with the existence of nonsense channels. We have discussed how Reggeism can be used to eliminate the ambiguities in choosing input forces for N/D type dynamical methods, and shown that for spinning problems the "input" and "output" Born terms are not identical.

Next we showed that popular versions of the N/D method were not, in general, consistent with Reggeism, but could be made so by choosing subtractions in a special way; we illustrated this in detail with the scalar-vector N/D method near $j=0$, showing how the new trajectory $\alpha_0(s)$ emerged naturally from such a computation. Finally, we discussed the bootstrap of PS particles from PS$-V$ scattering and obtained in lowest order simply polynomial equations whose coefficients were essentially spinological functions. The most plausible interpretation of the physical meaning of this type of approximation is that it has none, and is rather a sign of the basic weakness of the N/D method.